Elliptic CY-3 folds, BPS state counting and weak Jacobi-forms

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❶ Topological String Theory:

Benchmark problem: Solve the topological String on compact Calabi-Yau 3 folds $\leftrightarrow$ Determine the closed string partition function.

String theory is defined by map

$$x : \Sigma_g \to M \times \mathbb{R}_{3,1}$$

from a 2d world-sheet Σ_g of genus g into a target space $M \times \mathbb{R}_{3,1}$. Σ_g is equipped with a 2d super diffeomorphism invariant action $S_B(x, h, \phi, M)$ of type II. The partition function of the first quantized string is formally

$$Z(G,B)=\int \mathcal{D}x\mathcal{D}h\mathcal{D}\mathrm{ferm}\; e^{\frac{i}{\hbar}S(x,h,\phi,\mathrm{ferm},G,B)} \;.$$

By super symmetric localisation the integral localizes to $\delta S_B=0$, i.e. in the A model maps with minimal area called (j,J) holomophic maps x_{hol}, so that

$$\int \mathcal{D}x\mathcal{D}h \rightarrow \sum_{g,\kappa\in H_2(M,\mathbb{Z})}\int_{\mathcal{M}_{g,\kappa}} c^{vir}_{g,\kappa}=\sum_{g,\kappa\in H_2(M,\mathbb{Z})} r^{\kappa}_g$$

becomes a discrete sum

$$Z_{GW}(G, B) = \exp(F) = \exp\left(\sum_{g=0}^{\infty} {g_s}^{2g-2} F_g(\underline{z(\underline{t})})\right)$$

$$F_g = \sum_{\kappa \in H_2(M,\mathbb{Z})} r_g^{\kappa} Q^{\kappa} \ .$$

$$x : \Sigma_g \to M \qquad \begin{cases} \kappa & = [x(\Sigma_g)] \in H_2(M, \mathbb{Z}) \quad \text{class of image} \\ g & \quad \text{genus of WS} \end{cases}$$

Here $Q^{\kappa} = \exp(2\pi i \sum_a t_a \kappa^a)$ with $t_a = \int_{[\mathcal{C}_a]} (B + iJ)$ depend on the background.

1.) Solving the benchmark problem determines all the essential topological invariants of that map the Gromov-Witten invariants r_g^κ.

2.) It solves also a sheaf counting problem: Pandharipande-Thomas invaraints of stable pairs.

- F pure sheaf of dimension one. $\mathrm{ch}_2(\mathcal{F}) = \kappa$. D2-brane

- $s \in H^0(\mathcal{F})$ generates $\mathcal{F}$ outside $\chi(\mathcal{F}) = n$ points. D0-brane

$\rightarrow$ complex

$$I^\bullet : \mathcal{O}_M \xrightarrow{s} \mathcal{F}$$

From Serre duality:

$$\underset{\text{def}}{\mathrm{Ext}^1(I^\bullet, I^\bullet)} \sim \underset{\text{obs}}{\mathrm{Ext}^2(I^\bullet, I^\bullet)}$$

follows that the moduli space of stable pairs $\mathcal{P}_n(M, \kappa)$ supports a perfect obstruction theory.

With the definition: $P_{n,\kappa} \sim \text{degree of fundamental class}$ one has

$$Z_{PT} = \sum_{n\kappa} P_{n\kappa} \left(-e^{ig_s}\right)^n Q^\kappa = Z_{GW}(Q, g_s)$$

3.) It solves 5d BPS counting problems: Using ideas of heterotic/Type II duality Gopakumar and Vafa found

$$F(g_s, t) = \sum_{\substack{g\geq 0 \\ m\geq 1 \beta\in H_2}} \frac{I_g^\kappa}{m}\left(2\sin\frac{m g_s}{2}\right)^{2g-2} Q^{\kappa m} \ .$$

The $I_g^\kappa \in \mathbb{Z}$ are indices of BPS states with charge $\kappa \in H_2(M, \mathbb{Z})$ and a spin representation (j^L, j^R) in the little group $su_l(2) \times su_r(2)$ of the five dimensional

Lorentz group

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}}(-)^{2j_3^R} u^{2j_3^L} q^H = \sum_{\kappa} \sum_{g \in \mathbb{Z}_{\geq 0}} I_g^{\kappa} \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)^{2g} q^{\kappa}$$

In geometries, which admit additional $\mathbb{C}^*$ isometries, one can refine the index with $[j]_x := x^{-2j} + x^{-2j+2} + \ldots + x^{2j-2} + x^{2j}$ to $N^{\kappa}_{j_L j_R}$

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS}}} u^{2j_3^L} v^{2j_3^R} q^H = \sum_{\kappa} \sum_{j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} N^{\kappa}_{j_L j_R} [j_L]_u [j_R]_v q^{\kappa} \ ,$$

where $N^{\kappa}_{j_L j_R} \in \mathbb{N}$ are interpreted as *dimensions* of BPS representations.

Example 1: Huang, Poretschkin, AK: 1308.0619 local del Pezzo Calabi-Yau space $\mathcal{O}(-K_{d_8\mathbb{P}^2}) \to d_8\mathbb{P}^2$.

For the BPS states $N^{\kappa}_{j_l,j_r}$ at $\kappa = 2$ one gets:

$2j_l \backslash 2j_r$	0	1	2	3
0		3876		
1			248	
2				1

$\kappa = 2$

It is obvious that the adjoint represention $\mathbf{248}$ of E_8 appears as the spin $N^1_{\frac{1}{2},\frac{3}{2}}$, which decomposes into two Weyl orbits with the weights $w_1 + 8w_0$, further

$3876 = \mathbf{1} + \mathbf{3875}$, where the latter decomposes in the Weyl orbits of $w_1 + 7w_8 + 35w_0$.

Example 2: Katz, Pandharipande, AK: 1407.3181 $S = K3$ For the BPS states $N^d_{j_l,j_r}$ at $d = 3$ one gets:

$2j_L \backslash 2j_R$	0	1	2	3
0	1981		1	
1		252		
2	1		21	
3				1

$d = 3$

Now $1981 = 2 \cdot \mathbf{990} + \mathbf{1}$ and $\mathbf{252}$ seems to representations of the Mathieu group $M_{24} \in S_{24}$, which is one of sporadic finite groups.

4.) **Physics:** These geometric invariants determine parts of the spectrum of string, M- and F-theory compactifications. A direct physical motivation is to calculate the BPS saturated correlations functions in the effective 4d (6d) $N = 2$ field theory $F := F_0$

➪ gauge coupl: $g^{-2}_{IJ} = \mathrm{Im}\left(\bar{F}_{IJ} + \frac{2i\mathrm{Im}F_{IK}\mathrm{Im}F_{IL}X^K X^L}{\mathrm{Im}F_{KL}X^K X^L}\right)$

➪ BPS masses: $M^2_{n_E,n_M} = e^K \left|n_E t_E + n_M F_M\right|^2$

➪ grav couplings: $\int_{\mathrm{d}} x^4 F^g(t,\bar{t}) F_+^{2g-2} R_+^2$.

❷ The result for elliptically fibred CY- 3folds:

Motivation: Non-compact toric manifolds M_{nc}

$$\mathcal{O}(K_B) \to M_{nc} \to B \ \text{ toric}$$

Here the propblem can be solved using

- Localization Klemm Zaslow 99: Athiyah-Bot $(\mathbb{C}^*)^r$ equivariant loc.
- Large N-duality Aganagic, Klemm, Marino, Vafa 03: Topological Vertex

- Matrix model Bouchard, Klemm, Marino, Pasquetti 07: Eynard Orantin recursion

Now consider the simplest compactification of the local models described above: Let M be an elliptically fibred 3-fold over a 2d (Fano) surface B

$$\mathcal{E} \longrightarrow M \longrightarrow B$$

To make the formulas concrete we consider here the simplest case that $\mathcal{E}$ is an elliptic fibration with one global section and at codim one only singularities of Kodaira type I_1. The case with more sections was addressed in the introduction. Singular fibres have been also discussed in certain cases. E.g. Kodaira fibre I_0^* in Haghighat, Lockhardt, Vafa, A.K. 1412.3152.

By the Leray Serre spectral sequence $H_4(M)$ split into

$$\begin{array}{cc} divisor & dual\ curve \\ [D_e] & [\mathcal{C}^e] \\ & \\ [D_k], k = 1, \ldots, b_2(M) & [\mathcal{C}^k] \\ Kahler\ cone & Mori\ cone \end{array}$$

With the definitions:

$$a_k = K_B \cdot \check{D}_k, \quad a = K_B \cdot \mathcal{C}^k, \quad c_{ij} = \check{D}_i \cdot \check{D}_j$$

we have

$$D_e^3 = \int_B c_1^2(T_B), \quad D_e^2 D_k = a_k, \quad D_e D_i D_j = c_{ij}, \quad D_i D_j D_k = 0$$

$$c_2(T_M) \cdot D_e = \int_B 11 c_1^2 + c_2, \;\; c_2(T_M) \cdot D_k = 12 a_k, \;\; e = -60 \int_B c_1^2(T_B)$$

Note for

$$\tilde{\mathcal{C}}^k = \mathcal{C}^k + a^k \mathcal{C}^e, \qquad \tilde{D}_e^2 \tilde{D}_k = 0 \ .$$

We denote τ and T_k, $k = 1, \ldots, b_2(B)$ be the Kähler parameters of the elliptic fiber $\mathcal{E}$ and the base respectively. Further $q = \exp(2\pi i \tau)$ and $Q^\beta = \exp(2\pi i \sum_k T_k \beta_k)$.

Let us expand Z in terms of the base degrees β as

$$Z(\underline{t}, g_s) = Z_0(\tau, \lambda) \left(1 + \sum_{\beta \in H_2(B,\mathbb{Z})}^{\infty} Z_\beta(\tau, g_s) Q^\beta \right) \quad .$$

Property 1: The $Z_\beta(\tau, g_s)$ are meromorphic Jacobi-forms

$$\text{of} \quad \text{weight} = 0, \quad \text{and} \quad \text{index} = \beta \cdot (\beta - K_B) \ .$$

The poles are only at the torsion points of the elliptic argument

$$g_s = 2\pi i z,$$

which is indentified with the topological string coupling!

Property 2: The $Z_{\beta>0}(\tau, g_s)$ are quotients of *weak* Jacobi-forms of the form

$$\boxed{Z_\beta = \frac{1}{\eta^{12\beta\cdot K_B}} \frac{\varphi_\beta(\tau, z)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \varphi_{-2,1}(\tau, sz)}}$$

where $\varphi_\beta(\tau, z)$ is a weak Jacobi form of weight

$$k_\beta = 6\beta \cdot K_B - 2 \sum_{l=1}^{b_2(B)} \beta_l$$

and index

$$m_\beta = \frac{1}{6} \sum_{l=1}^{b_2(B)} \beta_l (1 + \beta_l)(1 + 2\beta_l) - \frac{1}{2}\beta \cdot (\beta - K_B) \ .$$

Inspired from many sources. E.g. Yau-Zaslow type formulas and formulas for the elliptic genus of quiver theories by Haghighat, Lockardt and Vafa.

Property 3:

Genus zero information and the Castelnovo bounds $I_g^\kappa = 0$ for $g > \mathcal{O}(\kappa^2)$ are sufficient to fix the mero– morphic Jacobiforms $\phi_\beta(\tau, z)$ if $\beta \cdot (\beta - K_B) \leq 0$.

Application: Hirzebruch surface $\mathbb{F}_1$

Take as B the Hirzebruch surface $\mathbb{F}_1$. This a rational fibration with a (-1) curve as the section. Together with the elliptic fibre M contains the elliptic surface $\frac{1}{2}K3$ with 12 I_1 fibres, which gives rise to the E-string over the (-1) curve as well as an elliptic K3 over the (0) curve.

$\beta = (b_1, b_2) \in H_2(F_1, Z)$, b_1 the degree w.r.t. to the (-1) section b_2 the degree w.r.t. the fibre.

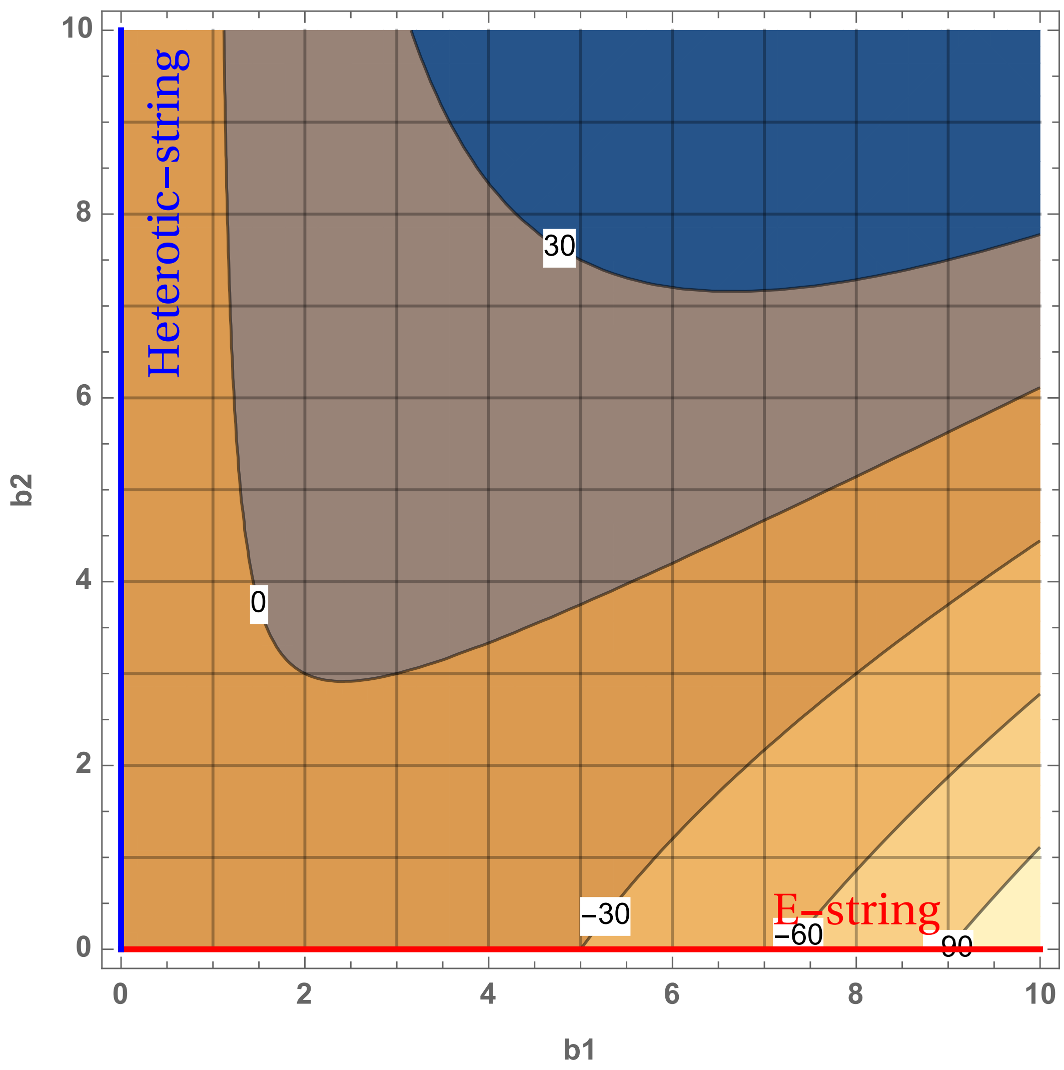

Figure 1: $\beta(\beta - K_B)$ in the Kähler moduli space of $\mathbb{F}_1$.

This unifies and extends many results. E.g. the heterotic oneloop calculations by Mariño and Moore 9808131 or the calculation of the E-string of Kim, Kim, Lee, Park and Vafa 1411.2324 . Especially compared to the latter calculation it is much more efficient. One just need to solve linear equations. E.g. the E-string seven times wrapping the base has BPS states

$g\backslash d_E$ 0	7	8
0	744530011302420	302179608949887585
1	-2232321201926990	-1227170805326730120
2	3903792161941380	2934388852145677599
3	-5068009339151240	-5282684497596522786
4	5291345197108229	7778874714012336871
5	-4601628396045684	-9724666039599532834
6	3391929155768781	10524550931465032549
7	-2138001602237932	-9971103737159845058
8	1156878805588608	8324325929288612251
9	-537744494290146	-6147084001181117522
10	214351035975405	4023020418703585279
11	-72999559484682	-2334951858562249752
12	21120665875714	1201406036917124067
13	-5151342670818	-547355661903552212
14	1048275845102	220379503469137845
15	-175554017242	-78203056459590866
16	23750162496	24372956004203707
17	-2529356130	-6642133492228324
18	204185633	1574080406463797
19	-11773768	-322162302125714
20	436550	56453421286247
21	-8246	-8376982135660
22	29	1037682979689

☆ Jacobi forms

❶ Definition of Jacobi forms

Jacobi forms $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group (Eichler & Zagier)

$$\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \; z \mapsto z_\gamma = \frac{z}{c\tau + d} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2;\mathbb{Z}) =: \Gamma_0$$

as

$$\varphi\left(\tau_\gamma, z_\gamma\right) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z)$$

and under quasi periodicity in the elliptic parameter as

$$\varphi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau, z), \quad \forall \quad \lambda, \mu \in \mathbb{Z} \,.$$

Here $k \in \mathbb{Z}$ is called the *weight* and $Bm \in \mathbb{Z}_{>0}$ is called the *index* of the Jacobi form.

The Jacobi forms have a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r} c(n,r) q^n y^r, \qquad \text{where } q = e^{2\pi i\tau}, \;\; y = e^{2\pi i z}$$

Because of the quasi peridicity one has $c(n,r) =: C(4nm - r^2, r)$, which depends on r only modulo $2m$. For a holomorphic Jacobi form $c(n,r)=0$ unless $4mn \geq r^2$, for cusp forms $c(n,r)=0$ unless $4mn > r^2$, while for weak Jacobi forms one has only the condition $c(n,r)=0$ unless $n \geq 0$.

❷ The ring of weak Jacobi forms

A weak Jacobi form of given index m and even modular weight k is freely generated over the ring of modular forms of level one, i.e. polynomials in $Q = E_4(\tau)$, $R = E_6(\tau)$ and $A = \varphi_{0,1}(\tau, z)$, $B = \varphi_{-2,1}(\tau, z)$ as

$$J^{weak}_{k,m} = \bigoplus_{j=0}^{m} M_{k+2j}(\Gamma_0)\varphi^{j}_{-2,1}\varphi^{m-j}_{0,1} \ .$$

The generators are the Eisenstein series E_4, E_6

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau + n)^k} = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \ ,$$

as well as

$$A = -\frac{\theta_1(\tau,z)^2}{\eta^6(\tau)}, \qquad B = 4\left(\frac{\theta_2(\tau,z)^2}{\theta_2(0,\tau)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(0,\tau)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(0,\tau)^2}\right) \ .$$

To summarize generators and quantitites defining the tpological string partition function

	Q	R	A	B	φ_b	$Z_b(\tau,z)$
weight k:	4	6	-2	0	$16b$	0
index m:	0	0	1	1	$\frac{1}{3}b(b-1)(b+4)$	$\frac{b(b-3)}{2}$

Since the numerator in

$$Z_b(\tau,z) = \frac{\varphi_b(\tau,z)}{\eta^{36b}(\tau)\prod_{k=1}^{b}\varphi_{-2,1}(\tau,kz)}.$$

is finitly generated, we can get for each b the full genus answer based on a finite number of boundary data

- the conifold gap condition and reguarity at the orbifold Huang, Quakenbush, A.K. hep-th/0612125

- the involution symmetry on $\mathcal{M}$ $I:\Omega\mapsto i\Omega$ $\leftrightarrow$ fibre modularity

- the parametrization of Z in terms of weak Jacobi-Forms

we can solve the compact elliptic fibration over $\mathbb{P}^2$ to $b=20$ for all d_E and $\forall\, g$ or to genus 189 $\forall\, b$ and $\forall\, d_E$.

❸ Witten's wave function and weak Jacobi-forms

Witten gave a wave function interpretation the topological string partition function, which implies

$$\left(\frac{\partial}{\partial(t')^{\bar a}}+\frac{i}{2}g_s^2 C_{\bar a}^{bc}\frac{D}{Dt^b}\frac{D}{Dt^c}\right)Z(g_s,\tau,b)=0\ ,$$

and summarizes all holomorphic anomaly equations. We want to study this in limit of large base B. The topological data fix the Kähler K potential the and Weil

Peterssen metric $G^{i\bar{j}}$ via the prepoterntial

$$F^{(0)} \sim -\frac{\kappa_{abc}}{3} t^a t^b t^c + \chi(M) \frac{\zeta(3)}{2(2\pi i)^3} + \sum_{\beta \in H_2(M,Z)} n_0^{\beta} \mathrm{Li}_3(q^{\beta}).$$

Now analyze $C_{\bar{a}}^{bc} := e^{2K} c_{\bar{a}\bar{b}\bar{c}} G^{b\bar{b}} G^{c\bar{c}}$ in the limit $\mathrm{Im}(T) = T_i \to \infty$

$$e^{2K} = -\frac{1}{16\tau_2^2 T_i^4} + \mathcal{O}(\frac{1}{T_i^5}),$$

$$C^{ij}_{\bar\tau} = \begin{pmatrix} -\frac{2\tau_2^2 h^4}{V} + \mathcal{O}(h^5) & A^1 h^3 + \mathcal{O}(h^5) & \dots & A^{r-1} h^3 \mathcal{O}(h^5) \\ A^1 h^3 + \mathcal{O}(h^5) & & & \\ \vdots & & -\frac{1}{4\tau_2^2} c^{kl} + \mathcal{O}(h) & \\ A^{r-1} h^3 + \mathcal{O}(h^5) & & & \end{pmatrix}$$

Applied to the wave function equation of Z with $(t')^{\bar a} = \bar\tau$ and $Q^\beta = e^{2\pi i b T}$, we get in the large base limit, because of the special from of the intersection matrix of elliptically fibered Calabi-Yau 3 folds only derivatives in the base direction T^i for t^b and t^c.

Identifying g_s with $2\pi i z$ and using the fact that the only

$\bar{\tau}$ dependence is in $\hat{E}_2$ this becomes

$$\boxed{\left(\partial_{\hat{E}_2} + \frac{\beta \cdot (\beta - K_B)}{24} z^2\right) Z_\beta(\tau, z) = 0}$$

which is solved by a weak Jacobi form of index $m = \frac{b(b-3)}{2}$ as we argue below.

Because of modularity and quasiperiodicity given a weak Jacobi form $\varphi_{k,m}(\tau, z)$ one can always define modular form of weight k as follows

$$\tilde{\varphi}_k(\tau, z) = e^{\frac{\pi^2}{3} m z^2 E_2(\tau)} \varphi_{k,m}(\tau, z)\ .$$

It follows that the weak Jacobi forms $\varphi_{k,m}(\tau,z)$ have a Taylor expansion in z with coefficients that are quasi-modular forms as in Eichler and Zagier [1].

$$\varphi_{k,m} = \xi_0(\tau)+\left(\frac{\xi_0(\tau)}{2}+\frac{m\xi_0'(\tau)}{k}\right)z^2+\left(\frac{\xi_2(\tau)}{24}+\frac{m\xi_1'(\tau)}{2(k+2)}+\frac{m^2\xi_0''(\tau)}{2k(k+1)}\right)z^4+\mathcal{O}(z^6)\ .$$

Moreover one has

$$\left(\partial_{E_2}+\frac{mg_s^2}{12}\right)\varphi_{k,m}(\tau,z)=0\ .$$

In particular A and B are quasi-modular forms that

[1] E.g. $\phi_{-2,1}(\tau,z) = -z^2+\frac{E_2 z^4}{12}+\frac{-5E_2^2+E_4}{1440}z^6+\frac{35E_2^3-21E_2E_4+4E_6}{362880}z^8+\mathcal{O}(z^{10})$.

satisfy the modular anomaly equation

$$\partial_{E_2} A = -\frac{g_s^2}{12} A, \qquad \partial_{E_2} B = -\frac{g_s^2}{12} B \ . \tag{1}$$

We can write this as the holomorphic anomaly equation

$$\left(2\pi i \mathrm{Im}^2(\tau) \bar{\partial}_{\bar{\tau}} - \frac{m g_s^2}{4} \right) \hat{\varphi}_{k,m}(\tau, z) = 0 \ . \tag{2}$$

☆ Compact elliptically fibred CY- manifolds

❶ Global fibration over $\mathbb{P}^2$

The formalism leads to a series of all genus predictions of BPS invariants for low base degree HKK'15. E.g. for $b=1$ and $b=2$ the numerator is

$$\varphi_1 = -\frac{Q(31Q^3+113P^2)}{48},$$

which leads to the following prediction of BPS invariants

$g\backslash d_E$	$d_E=0$	1	2	3	4	5	6
$g=0$	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398
2	0	0	9	-3192	412965	614459160	68590330119
3	0	0	0	-12	4230	-541440	-820457286
4	0	0	0	0	15	-5256	665745
5	0	0	0	0	0	-18	6270
6	0	0	0	0	0	0	21

Table 1: Some BPS invariants $n^g_{(d_E,1)}$ for base degree $b=1$ and $g, d_E \leq 6$.

$$\begin{aligned}\varphi_2 \quad = \quad & \frac{B^4Q^2\left(31Q^3+113R^2\right)^2}{23887872}+\frac{1}{1146617856}[2507892B^3AQ^7R+9070872B^3AQ^4R^3 \\ & +2355828B^3AQR^5+36469B^2A^2Q^9+764613B^2A^2Q^6R^2-823017B^2A^2Q^3R^4 \\ & +21935B^2A^2R^6-9004644BA^3Q^8R-30250296BA^3Q^5R^3-6530148BA^3Q^2R^5 \\ & +31A^4Q^{10}+5986623A^4Q^7R^2+19960101A^4Q^4R^4+4908413A^4QR^6]\ , \qquad (3)\end{aligned}$$

$g\backslash d_E$	$d_E=0$	1	2	3	4	5	6
$g=0$	6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072
1	0	15	-8574	2126358	521856996	1122213103092	879831736511916
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672
3	0	0	0	66	-45729	627574428	3776946955338
4	0	0	0	0	-132	-453960	-95306132778
5	0	0	0	0	0	-5031	1028427996
6	0	0	0	0	0	-18	-771642
7	0	0	0	0	0	0	-7224
8	0	0	0	0	0	0	-24

Table 2: Some BPS invariants for $n^g_{(d_E,2)}$

❷ Checks form algebraic geometry:

Using the definition of BPS states as Hodge numbers of the BPS moduli space, we get vanishing conditions, from the Castelnouvo bounds, as well as explicite results for non singular moduli spaces:

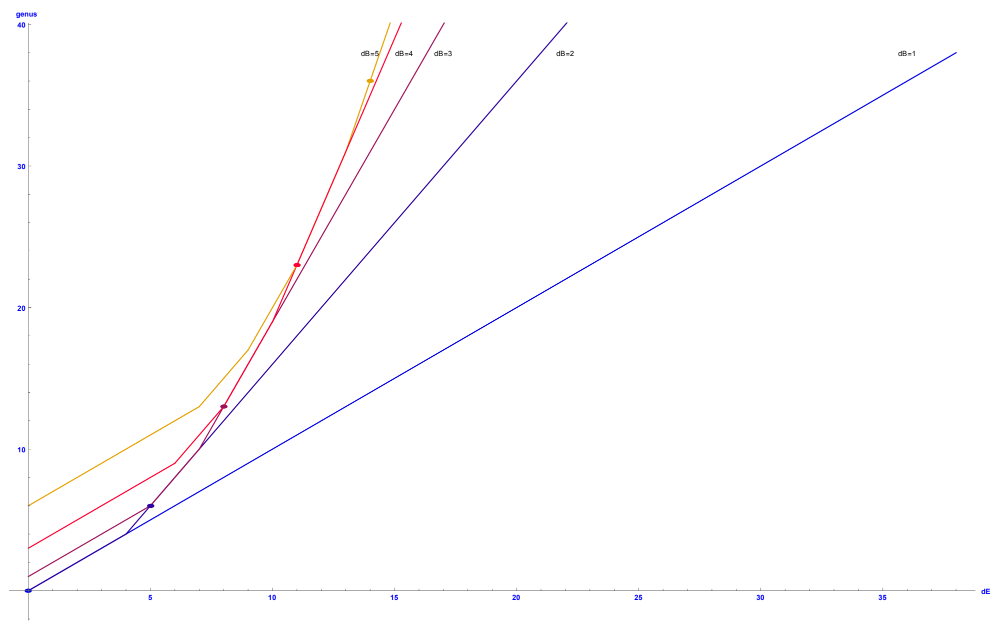

Figure 2: The figure shows the boundary of non-vanishing curves for the values of $b = 1, 2, 3, 4, 5$.

Computing the Euler characteristic of the BPS moduli space, we obtain for these values on the edges of the figure

$$n_{d_E,b}^{d_Eb-(3b^2-b-2)/2} = (-1)^{d_Eb-(1/2)(3b^2+b-4)} \\ 3\left(d_Eb - \frac{3b^2+b-6}{2}\right) .$$

which perfectly matches the predction of the weak Jacobi forms.

☆ Conclusions:

$$\boxed{Z_\beta = \frac{1}{\eta^{12\beta\cdot K_B}} \frac{\varphi_\beta(\tau, z)}{\prod_{l=1}^{b_2(B)} \prod_{s=1}^{\beta_l} \varphi_{-2,1}(\tau, sz)}}$$

➨ Since the elliptic argument z of the Jacobi forms is identified with the string coupling

$$g_s = 2\pi i z$$

this expression captures all genus contributions for a given base class.

➼ From the transformation properties of weak Jacobi-forms it follows that the depence of Z on string the coupling is coupling is quasi periodic.

➼ Since (1) has poles only at the torsion points of the elliptic argument

$$Z_b(\tau, z) = Z_b^{pol} + Z_b^{fin},$$

where the finite part

$$Z_\beta^{fin}(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau)\theta_{m,l}(\tau, z)$$

has an expansion in terms of mock modular forms $h_l(\tau)$.

- The latter fact can be used to check the microscopic entropy of 5d $N = 2$ spinning black holes and the wall crossing behaviour of 4d BPS states. Some partial results have been obtained by Vafa et. al. arXiv:1509.00455

- We can make infinitly many checks from algebraic geometry for those curves which have smooth moduli spaces, as seen above. But e.g. for $b = 1$ one can confirm the formulas for all classes Jim Bryan et. all work in progress

- We can solve the E-string complety. There are good indications that the more gerenal decoupling criteria for general 6d theories are similar strong as the condition $\beta \cdot (\beta - K_B) \leq 0$. This might lead to general formulas for the elliptic genera of $6d$ theories.

- The geometries are the most natural compactification of the local toric geometries that we can solve with very interesting methods: Localization, vertex, matrix model. This begs for an extension of these techniques. E.g. an elliptic vertex which solves the compact Calabi-Yau cases.